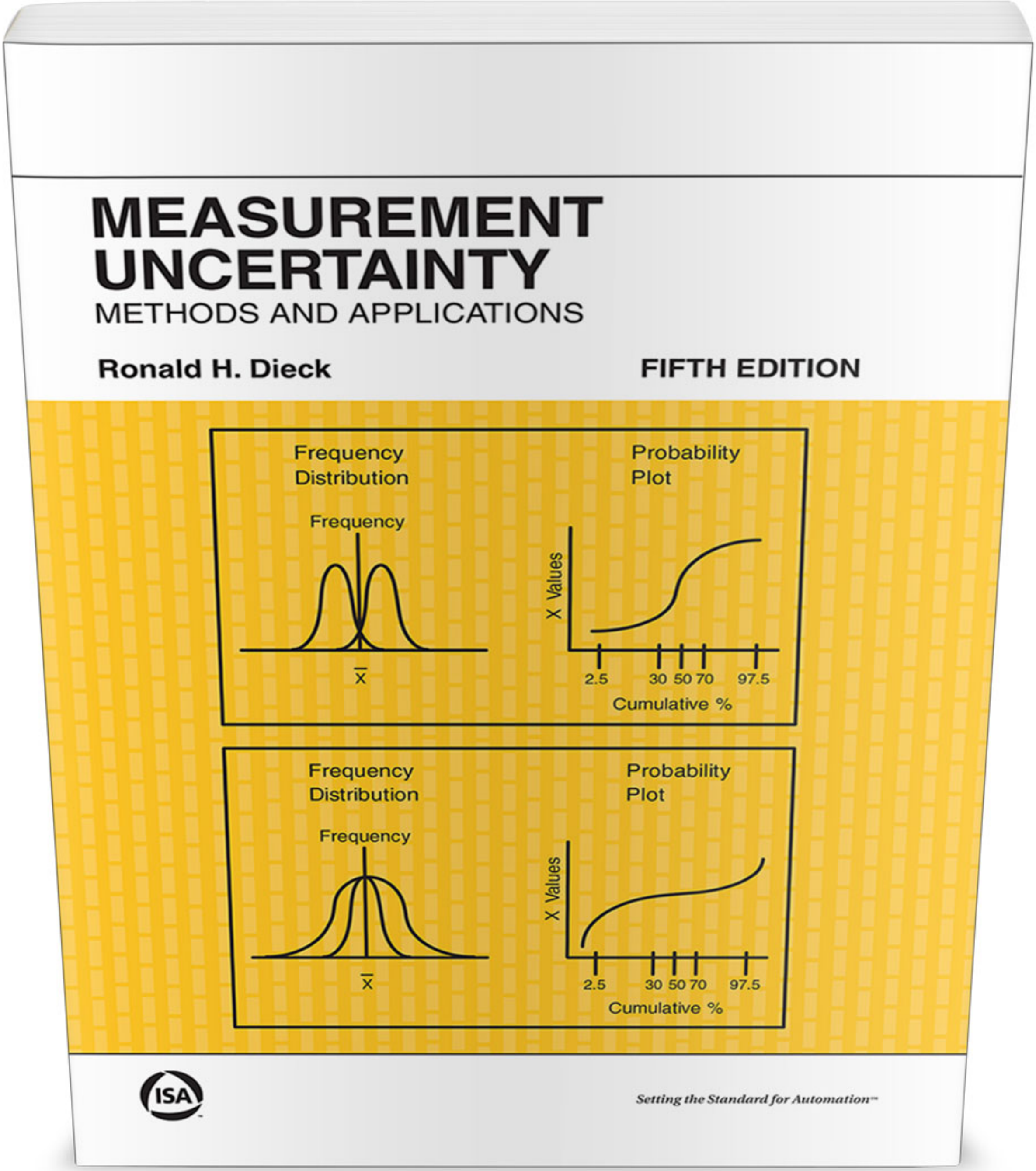
MEASUREMENT
UNCERTAINTY
METHODS AND APPLICATIONS
Ronald H. Dieck
FIFTH EDITION
Frequency
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Plot
X Values
2.5
30 50 70
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Measurement Uncertainty:
Methods and Applications
Fifth Edition

by
Ronald H. Dieck

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Unit 3:
The Measurement Uncertainty Model

UNIT 3

The Measurement Uncertainty Model

In this unit, procedures and models will be developed for providing decision makers with a clear, unambiguous statement of the accuracy or uncertainty of their data. No manager should ever be without a statement of measurement uncertainty attendant to each piece of data on which decisions are based. No experimenter should permit management to consider measurement data without also considering its measurement uncertainty. The manager has the responsibility for requiring measurement uncertainty statements. The experimenter has the responsibility for never reporting test results without also reporting their measurement uncertainty.

Learning Objectives—When you have completed this unit you should:

1. **Understand the need for a single-valued uncertainty statement.**
2. **Be able to characterize error sources as systematic or random.**
3. **Be able to combine elemental systematic and elemental random uncertainties into the systematic uncertainty and random uncertainty for the measurement process.**
4. **Be able to combine the process systematic and random uncertainties into a single value of measurement uncertainty for the measurement process.**
5. **Recognize that the ISO (Type A and B) uncertainty analysis yields the same 95% confidence uncertainty as the ASME (random and systematic) uncertainty analysis, although the ASME method is recommended by this text.**

3-1. The Statement of Measurement Uncertainty

The purpose of measurements is to numerically characterize the state or performance of a physical process. Properly understanding the data from measurements requires a statement of uncertainty.

In the previous unit, the concepts of systematic error (and uncertainty) and random error (and uncertainty) were developed. (ISO Types A and B uncertainties were also discussed.) It is not enough to be able to state the systematic and random uncertainties of a measurement. Management decision makers do *not* want to hear a flow level is 3000 scfm with ±30 scfm systematic uncertainty, ±14 scfm standard deviation of the average with 75 degrees of freedom. They want a single number from which to understand the limits of accuracy for the 3000 scfm measurement. That single number is the statement of *measurement uncertainty*. It is a single, unambiguous number. It is an objective estimate of the data quality.

3-2. Grouping and Categorizing Error Sources

To conduct a proper uncertainty analysis, it is necessary to identify the error sources that affect an experimental result and to characterize them as systematic or random. ISO's guide (Ref. 1) recommends grouping error sources and uncertainties according to the origin of their estimation. The guide recommends using two types of uncertainties: Type A, for which there is data to calculate a standard deviation, and Type B, for which there is not.

This kind of Type A and Type B characterization is not required for uncertainty analysis. However, it is recommended that the uncertainty "type" as noted by ISO be used. The easiest way to indicate the origin, or pedigree, of the uncertainty estimating data is to add a subscript to each elemental uncertainty estimate. In this way, the engineering grouping of systematic and random may be retained, as well as the ISO grouping by information source. This compromise is recommended in the new, national standard on measurement uncertainty (Ref. 2). (See Appendix L for a detailed discussion of the advantages and disadvantages of the ISO, ASME, and Basic uncertainty models.)

In this book, where not otherwise noted, it will be assumed that all systematic uncertainties are Type B and all random ones are Type A.

In addition to the above, the bookkeeping process is eased when the error sources are grouped into categories (Ref. 3). It is instructive to group error sources into four categories: calibration, data acquisition, data reduction, and errors of method. However, this grouping of error sources (also uncertainty estimates) is not needed for an uncertainty analysis.

Grouping Error Sources

Calibration errors are those that result from the laboratory certification of an instrument response. Usually, this category includes errors that result from the use of instruments at the test site (sometimes called *installation errors*). It is very important to note that these are not the usual calibration errors that are calibrated out by the calibration process. For example, it may be known that a pressure transducer will have an uncertainty of ±2% if it is not calibrated and ±0.5% when it is. In this case, the large known systematic uncertainty (±2%) has been traded for the smaller uncertainties in the calibration process (±0.5%). Uncertainty sources totaling ±1.5% have been removed. What remains is the unknown error of the calibration process (which is estimated by an uncertainty). Only its limits are known: ±0.5%. It is this ±0.5% that will enter the uncertainty analysis (properly split between systematic and random uncertainty). These uncertainties are determined by a careful uncertainty analysis of the instrument calibration and use process.

Data acquisition errors are those that result from the use of data acquisition equipment, which may be computer data acquisition systems, a data logger, or just a person with pencil and paper reading a meter. Acquiring data to estimate the magnitude of these error sources is usually simple. All that is needed is to take multiple readings over the appropriate time period. This category of errors is also usually smaller in magnitude than calibration errors by a factor of three to ten. As with calibration errors, the exact magnitude of these errors is not known; only their limits may be estimated: systematic and random uncertainties.

Data reduction errors are those that most often result from the use of computers or pocket calculators. Truncation, roundoff, and approximate solutions are examples in this category. These error sources are usually smaller than either of the former two sources, but they can be significant—most commonly due to improper curve fitting. This will be discussed in Unit 7.

Errors of method have to do with the ability of the experimenter to adjust instruments on the test site. These errors are sometimes called *personal errors*. Sometimes this group also includes *sampling error*, that is, how well a measurement system is characterized considering its own variability or profiles. Sampling error is discussed in Unit 7.

Grouping errors into categories is not a necessary step to a correct uncertainty analysis. It is only a bookkeeping aid. (It is fully correct to conduct an uncertainty analysis with only one big group for all the error sources, as is done in this text.)

Note: Units 3 and 4 present the calculation of measurement uncertainty for a single measurement or parameter, such as temperature or pressure. When these measurements are then used to calculate a result, such as flow, the systematic and random uncertainties must be propagated into that result, using the methods in Unit 5.

Categorizing Errors and/or Uncertainties

Early in an uncertainty analysis it is necessary to decide whether an error source is systematic or random. The defined measurement system will provide the information necessary to make this choice. There are times when the same error source is a systematic error in one circumstance and a random one in another.

For example, in a multifacility test, the scatter calculated between facilities would be a random uncertainty ($s_{\overline{X}}$) if one were describing facility-to-facility variability. However, from the perspective of working in only one facility, that same data scatter indicates that one facility could have a systematic error compared to the others by an amount within the interval of $s_{\overline{X}}$ (the systematic uncertainty). The exact same error source is on one occasion random and on another systematic, depending on the context of the test.

Remember, error is the actual true difference between the measured data and the true value. One never knows the true value or the true error, but can only estimate its limits. This is called the *uncertainty*. In estimating the limits of a systematic error, use systematic standard uncertainty. For random error, use random uncertainty.

The following is the best rule to follow: If an error source causes scatter in the test result, it is a *random error*. All other error sources are *systematic errors.* This is without regard to the source of data used to estimate an uncertainty. The emphasis here is on the effect of the error source. It does not matter where the error comes from; it matters where it's going—its effect. What matters is the effect that the error source has on the manager's ability to make decisions with the test results. Errors have the effect of inhibiting proper decisions or, worse, selling a faulty product. Systematic errors, while invisible in test data, will cause an entire test program to be off from the true value. Random errors, their effects always visible in the test data, will cause scatter in the test results—always, by definition.

Categorizing error sources, therefore, is simply utilizing the above rule. It works in every case. It ignores origins and pedigrees. It considers what most experimenters want to know: What does an error or uncertainty do to the data or test result? This view allows one to apply the knowledge to the decision-making process that follows an experiment. After the above categorization, and after uncertainty estimates are made for the previously mentioned error sources, subscripts should be added to the elemental uncertainty values to denote the origin of the uncertainty estimates.

Designating Error Sources

A subscripted notation for error sources is described in Ref. 3. The general notation for an elemental systematic standard uncertainty is b_{ij}, where the b_{ij} is the systematic standard uncertainty estimate for the ith elemental error source in the jth category. Similarly, the notation for an elemental random error source (standard deviation) is s_{ij}, where s_{ij} is the standard deviation of the data from the ith error source in the jth category. Note that each s_{ij} would be divided by the appropriate N_{ij} to obtain the applicable random standard uncertainty, $s_{\overline{X},ij}$. These i and j subscripts are not utilized in this book to designate error sources in categories.

A simpler notation will work just fine. Note each uncertainty source with only one subscript to identify its source. Double subscripts may be used if a characterization into major groups is desired, but it is unnecessary in uncertainty analysis. Just denote each elemental systematic uncertainty as b_i and each random uncertainty as $s_{\overline{X},i}$. The single subscript method i will be utilized throughout this book to designate error or uncertainty sources.

3-3. The Calculation of Measurement Uncertainty (Symmetrical Systematic Uncertainties)

Combining Random Uncertainties

It is informative to note that usually there is no need to determine the effect of combined random errors once a test has been run. The scatter in the data is exactly the right effect of all the random error sources! The standard deviation of that scatter divided by the square root of the number of data points averaged is the random uncertainty, ($s_X/\sqrt{N}$). However, before scarce dollars are spent on a test program, it is important to know whether or not the results will be useful. In this important case, it is necessary to do a pretest uncertainty analysis as discussed in Unit 4. In this section, the methods for combining the effects of several uncertainty sources will be delineated. This combination is a requirement for understanding the expected uncertainty of a test before it is run.

The Standard Deviation: The Elemental Random Uncertainty

The first level of random uncertainty is the elemental random uncertainty. This is the standard deviation of the data caused by each random error source. The equation for the standard deviation of the elemental random error source is:

$$s_{X,i} = \left[\frac{\sum_{k=1}^{N_i}(X_{i,k} - \overline{X}_i)^2}{N_i - 1}\right]^{1/2} \tag{3-1}$$

Note: The sum is over k where there are N_i values of $X_{i,k}$.

Here, $s_{X,i}$ is the standard deviation of elemental random error source i. The standard deviation is calculated exactly as in Equation 3-1. If this were the only random error source, the test data would exhibit scatter identical to that described by $s_{X,i}$.

However, there is always more than one error source, and it is necessary to combine their effects when estimating what the random uncertainty for an experiment might be. Having combined them, it is necessary to estimate the appropriate degrees of freedom. The first step in this combination is the calculation of the standard deviation of the average, the random standard uncertainty, for an error source.

The Random Standard Uncertainty for an Error Source

What matters most about an error source is its average effect for a particular experimental result. Seldom does one care what the actual data scatter might be. One is always concerned about its average effect. In fact, test results are almost always reported as the average for the data obtained. The effect on the average for a particular elemental *random uncertainty* can be expressed as:

$$s_{\overline{X}, i} = \frac{s_{X, i}}{\sqrt{N_i}} \tag{3-2}$$

where

$s_{\overline{X},i}$ = the random standard uncertainty (or standard error of the mean) for error source i

N_i = the number of data points averaged for error source i

This is an outgrowth of the central limit theorem (Ref. 4). What is being done here is to estimate the standard deviation of the average from the standard deviation of the data. Note that N_i may or may not be the same as the N_i used to calculate the standard deviation. This denominator N_i is always the number of data points that are in the average, or test result.

What is most important is the effect that an elemental random error source has on the average value. Stated another way, what would the standard deviation of a group of averages be for a particular error source? Usually, there is neither time nor money to run numerous experiments so that many averages can be obtained. It is necessary, therefore, to estimate what that distribution of averages would be; in other words, what the standard deviation of that group of averages would be. That estimate is achieved with Equation 3-2, where the standard deviation of the data is used to estimate the standard deviation of the average for a particular elemental random error source.

The Standard Deviation of the Average for the Result: The Random Standard Uncertainty

It is then the combined effect of the several random uncertainties on the average for the test or the test result that needs evaluation. That combined effect is determined by the root-sum-square as in Equation 3-3:

$$s_{\overline{X}, R} = \left[\sum_{i=1}^{N_i} (s_{\overline{X}, i})^2 \right]^{1/2} \tag{3-3}$$

where $s_{\overline{X},R}$ and $s_{\overline{X},i}$ have the same units. Note that the sum is over i. It is the $s_{\overline{X},i}$ values that are root-sum-squared, not the $s_{X,i}$ values. This is an important fact to remember throughout the remainder of this book. We will always root-sum-square $s_{\overline{X}}$ (s-sub-X-bar) values.

$s_{\overline{X}}$ is sometimes called the *standard deviation of the average for the result* and it is the *random standard uncertainty for the result*. It forms the basis for the random part of the *uncertainty statement*. Note that $t_{95}\, s_R$ could be called the *random uncertainty component of the uncertainty statement*.

Combining Degrees of Freedom

In order to utilize the correct t_{95} (Student's t) in an uncertainty statement, it is necessary to know the correct degrees of freedom, v_R, associated with the *random standard uncertainty for the result*, $s_{\overline{X},R}$. Each of the elemental random standard uncertainties, $s_{\overline{X},i}$, has its associated degrees of freedom, v_i. The same degrees of freedom, v_i, are associated with the corresponding standard deviation of the average for the error source, $s_{\overline{X},i}$. When the $s_{\overline{X},i}$ are combined with Equation 3-3, it is necessary to combine the degrees of freedom as well. Refs. 5 and 6 utilize the Welch-Satterthwaite approximation given originally in Ref. 7. There (in the absence of any systematic errors) the degrees of freedom for a test result are given as:

$$v_R = \frac{\left[\sum_{i=1}^{N} (s_{\overline{X},i})^2\right]^2}{\left[\sum_{i=1}^{N} \left(\frac{(s_{\overline{X},i})^4}{v_i}\right)\right]} \tag{3-4}$$

Remember:

$$s_{\overline{X},i} = \frac{s_{X,i}}{\sqrt{N_i}} \tag{3-5}$$

Remember too that:

$$s_{X,i} = \left[\frac{\sum_{i=1}^{N_i} (X_i - \overline{X})^2}{N_i - 1}\right]^{1/2} \tag{3-6}$$

Combining Symmetrical Systematic Standard Uncertainties

Systematic standard uncertainties may be expressed as *symmetrical*, the first case treated here, or *nonsymmetrical*, which will be presented later.

Systematic standard uncertainties come from various error sources, and their combined effect must be evaluated. This is true whether the effect is to be noted in a pretest uncertainty analysis or a post-test analysis. Remember, test data, while revealing the magnitude and effect of random errors, do not do so for systematic errors. One cannot see the effect of systematic errors in test data. It must be estimated from the combined effect of various sources.

The Symmetrical Systematic Standard Uncertainty for the Test Result

Each elemental systematic standard uncertainty, b_i , must have its combined effect on the test result evaluated. That combined effect is determined by Equation 3-7:

$$b_R = \left[\sum_{i=1}^{N} (b_i)^2 \right]^{1/2} \tag{3-7}$$

where b_R is the systematic uncertainty of the test result and where b_R and b_i have the same units.

b_R is the basis for the systematic standard uncertainty component of the uncertainty statement. It is the value used for the symmetrical systematic standard uncertainty. Note here that the systematic standard uncertainties, whether elemental or for the result, represent one standard deviation of the average as do the random standard uncertainties.

Summarizing Uncertainty Categorization (Symmetrical Systematic Standard Uncertainties)

Systematic and random error sources and the estimates of their limits, systematic and random standard uncertainties, and their categories may be summarized as shown in Table 3-1. Assuming that the values in Table 3-1 are the uncertainties at the temperature calibration point of interest, the systematic standard uncertainty for the calibration result is obtained as follows:

$$b_R = \left[\sum_{i=1}^{3}(b_i)^2\right]^{1/2} = [(0.025)^2 + (0.005)^2 + (0.01)^2]^{1/2} = 0.027 \quad (3\text{-}8)$$

To obtain the random standard uncertainty for this experiment, a similar rootsum-square approach is used as follows:

$$\begin{aligned} s_R &= [\sum(s_{X,i}/(N_i)^{1/2})^2]^{1/2} \\ &= [\sum(s_{\bar{X},i})^2]^{1/2} \\ &= [(0.056)^2 + (0.016)^2 + (0.045)^2]^{1/2} \\ &= 0.074°F \end{aligned} \quad (3\text{-}9)$$

Table 3-1. Temperature Calibration Uncertainties

Error Sources (i)	**°F Systematic Standard Uncertainty (b_i)**	**°F Standard Deviation ($s_{X,i}$)**	**No. Data Points Avg'd. (N_i)***	**°F Random Standard Uncertainty ($s_{\bar{X},i}$)**	**Degrees of Freedom (N – 1) (v_i)**
(1) Intermediate thermocouple reference	0.025_B	0.056_A	1	0.056_A	29
(2) Ice reference junction	0.005_B	0.016_A	1	0.016_A	9
(3) Voltmeter readout	0.01_B	0.1_A	5	0.045_A	4

*Number of data points averaged in this measurement. That is, 1 intermediate reference reading, 1 ice reference reading, and 5 voltmeter readings averaged for each temperature measurement.

Note: The degrees of freedom are associated with $s_{X,i}$ and $s_{\bar{X},i}$ and not N_i. Also, the subscripts A and B have been used to denote the existence or nonexistence of data to calculate the standard deviation for the uncertainty source shown. For all cases when an uncertainty is Type B, infinite degrees of freedom are assumed.

To compute the uncertainty, it is necessary to obtain a t_{95} value from Appendix D. To do that, the appropriate degrees of freedom must be obtained using the Welch-Satterthwaite approximation, Equation 3-4. Note that the systematic uncertainties, ISO Type B, must also be included, with degrees of freedom assumed to be infinite. This is because, you will recall, each b_i represents a 68% confidence estimate of a normal distribution of errors. So, each b_i represents one standard deviation.

$$v_R = \frac{[\sum(s_{\bar{X},i})^2 + \sum(b_i)^2]^2}{\left\{\sum[(s_{\bar{X},i})^4/v_i] + \sum[(b_i)^4/v_i]\right\}}$$

$$v_R = \frac{[(0.056)^2 + (0.016)^2 + (0.045)^2 + (0.025)^2 + (0.005)^2 + (0.01)^2]^2}{[(0.056)^4/29] + [(0.016)^4/9] + [(0.045)^4/4] + 0 + 0 + 0}$$
$$= 27.7 = 27$$

The three zeros in the denominator are a result of the degrees of freedom for the b_i terms being infinite.

Note that, to obtain the correct degrees of freedom, 27.7 is truncated to 27. Always truncate, as this is the conservative value for the degrees of freedom. The evaluation of why this is conservative is left as an exercise for the reader (see Exercise 3-5).

The table in Appendix D contains the number of degrees of freedom: $t_{95} = 2.052$ for 27 degrees of freedom.

The information is now available to form an uncertainty statement.

3-4. The Uncertainty Statement (Symmetrical Systematic Standard Uncertainties)

Measurement uncertainty is defined as the combination of both the systematic and random components of uncertainty, formerly called *bias* and *precision*. An appendix in Ref. 5 provides one method for that combination.

However, it is instructive to note two older methods that, while not now utilized, do appear in many books and papers. These older models are the *addition model* and the *root-sum-square model*. They are presented here for information only and will not be used in this book.

The Addition (ADD) Uncertainty Model

The addition uncertainty model is defined as:

$$U_{ADD} = \pm[B_R + t_{95}s_{\bar{X},R}] \tag{3-10}$$

where U_{ADD} is the additive uncertainty. Here B_R equals 2 * b_R.

U_{ADD} provides an interval (or coverage) around the test average that will contain the true value ~99% of the time (Ref. 8). It is also called U_{99} for that reason. Stated another way, the true value, μ, is contained in the interval:

$$(\bar{X} - U_{ADD}) \le \mu \le (\bar{X} + U_{ADD}) \qquad (3\text{-}11)$$

approximately 99% of the time. This interval may also be written as:

$$(\bar{X} - U_{99}) \le \mu \le (\bar{X} + U_{99}) \qquad (3\text{-}12)$$

Note:

$$U_{ADD} = U_{99} \qquad (3\text{-}13)$$

The Root-Sum-Square (RSS) Uncertainty Model

The root-sum-square uncertainty model is defined as:

$$U_{RSS} = \pm[(B_R)^2 + (s_{\bar{X},R})^2]^{1/2} \qquad (3\text{-}14)$$

where U_{RSS} is the root-sum-square uncertainty. Here B_R again equals $2 * b_R$.

U_{RSS} provides an interval around the test average that will contain the true value ~95% of the time (Ref. 8). It is also called U_{95} for that reason. Stated another way, the true value, μ, is contained in the interval:

$$(\bar{X} - U_{RSS}) \le \mu \le (\bar{X} + U_{RSS}) \qquad (3\text{-}15)$$

approximately 95% of the time.

However, Ref. 5 recommends newer methods, two improved uncertainty models: U_{95} and U_{ASME}. The origin and detailed development of these models are discussed in Refs. 1 and 2.

The U_{95} and U_{ASME} Uncertainty Models

In this book, the U_{ASME} model will be used in those cases where the degrees of freedom may be assumed to be 30 or higher. The U_{95} model will be used all other times. Note that U_{95} and U_{ASME} are both ASME models.

The U_{ASME} model is:

$$U_{ASME} = \pm 2.00[(b_R)^2 + (s_{\overline{X},R})^2]^{1/2} \quad (3\text{-}16)$$

This model provides 95% confidence under a vast variety of conditions (Ref. 9) and is the recommended model for easing uncertainty calculations. This is the case because the 2.00 in front of the equation is Student's t for 30+ degrees of freedom and 95% confidence. This simple model may also be altered to other confidences by using the U_{95} model:

$$U_{95} = \pm t_{95}[(b_R)^2 + (s_{\overline{X},R})^2]^{1/2} \quad (3\text{-}17)$$

Here there is no assumption of the degrees of freedom being 30 or higher, and an individual Student's t is calculated for each uncertainty statement. This model provides whatever confidence is desired by the experimenter. It does so by the proper selection of Student's t. That is, for 99% confidence and 30+ degrees of freedom, $t_{99} = 2.58$. For 99.7% confidence, $t_{99.7} = 3.0$. For any confidence and less than 30 degrees of freedom, a standard statistics text will contain the table from which Student's t can be obtained. The degrees of freedom for the U_{95} uncertainty model are obtained with the Welch-Satterthwaite approximation, in which the b_i are recognized as equivalent to one standard deviation of the average.

Although Equation 3-17 is a more robust uncertainty equation to use, the use of Equation 3-16 instead of 3-17 will seldom lead to difficulty in real world situations (Refs. 10 and 11).

It should be noted that some use the phrases *total uncertainty*, *expanded uncertainty*, and *combined uncertainty*. We will note only that *combined uncertainty* will be taken to mean the root-sum-square of the random standard uncertainty and the systematic standard uncertainty. The total uncertainty and expanded uncertainty will be considered synonymous and be taken to mean the *combined uncertainty* multiplied by the appropriate Student's t to obtain the 95% confidence interval.

Assumption of "b" Usage Throughout This Text

Note that in the above expressions for uncertainty, Equations 3-16 and 3-17, the systematic uncertainty is noted as a b term. This is because it is assumed that all the systematic standard uncertainties represent error sources that are normally distributed. In addition, the b term usage presumes that all the systematic standard uncertainties have infinite degrees of freedom, and are estimated at 68% confidence. This will be the assumption throughout this book unless otherwise noted.

Note now that this condition for the *b* terms although *most often* the case, will not *always* be the case in practice. There are times when systematic uncertainties, estimated at 68%, will not possess infinite degrees of freedom. A specific example of this would be the confidence one has in a calibration curve fit. In that case, the 95% confidence interval on the curve fit is estimated as:

$$Conf_{95} \approx t_{95} * (SEE)/\sqrt{N}$$

where N = the number of data points in the curve fit.

This $Conf_{95}$ is the systematic uncertainty of using the curve fit calculated line. When this term is incorporated into the calculation of the systematic uncertainty of the result, it should be entered as $Conf_{95}/t_{95}$. In this way, that term usually noted as b will maintain its equivalent 1s value for all root-sum-square operations to follow.

However, that being said, we will assume throughout this text that people make systematic uncertainty estimates at 95% confidence for normally distributed error sources with infinite degrees of freedom. That is, in the absence of any data, estimates of systematic uncertainty are typically approximately 95% confidence. This is taken as typical of individuals' estimates when they are requested to make those estimates without data. Other confidence intervals could be chosen but it is recognized that the most common occurrence is 95% confidence.

The U_{ISO} Uncertainty Model

Note that the calculation of uncertainty using the strict ISO approach yields the same result. In the strict ISO approach, we must evaluate U_A and U_B.

Using the example of results from Table 3-1, we note that $U_A = s_{\overline{X},R}$ and that $U_B = b_R$. It is b_R because all the b_i in Table 3-1 are 68% confidence estimates for a normal distribution and thus, equivalent 1s values.

Since $U_{ISO} = k[(U_A)^2 + (U_B)^2]^{1/2}$ there really is no difference in result between the U_{95} and the U_{ISO} where the ISO k multiplier is taken as Student's t. In addition, there is no difference between the U_{ASME} and the U_{ISO} where k is taken as 2.

3-5. Computing the Uncertainty Interval (Symmetrical Systematic Uncertainty)

The information is now available to compute the uncertainty interval, or measurement uncertainty, for the case in Table 3-1 with symmetrical systematic uncertainties. The values to be used are from Equations 3-8 and 3-9.

The U_{95} Uncertainty Interval (Symmetrical Systematic Uncertainty)

For the U_{95} uncertainty, the expression is:

$$U_{95} = \pm 2.052[(b_R)^2 + (s_{\bar{X},R})^2]^{1/2}$$
$$= \pm 2.052[(0.027)^2 + (0.073)^2]^{1/2}$$
$$= \pm 0.16\ °F$$

Assuming an average calibration correction of +2.00 °F at the temperature of interest, it can be stated that the interval:

$$(2.00 - 0.16) \leq \mu °F \leq (2.00 + 0.16)$$
$$1.84 \leq \mu °F \leq 2.16$$
$$\mu = 2.00 \pm 0.16 °F$$

contains the true value for the correction, μ, 95% of the time; that is, there is 95% confidence that the true value is contained in the above interval.

Note that the calibration constant, or correction, is not the uncertainty or the error. The uncertainty is now how well the calibration correction [+2.00F] is known [±0.016F].

The concept of confidence is shown schematically in Figure 3-1 for the U_{ASME} model (which is the easiest to show graphically). Of importance to note is that the uncertainty interval is widened by both the random and the systematic components of uncertainty. In Figure 3-2, the total (~68% confidence not expanded) uncertainty interval of 2.00 ± 0.16 is shown; in this interval, the true value of the temperature calibration correction will be contained 95% of the time.

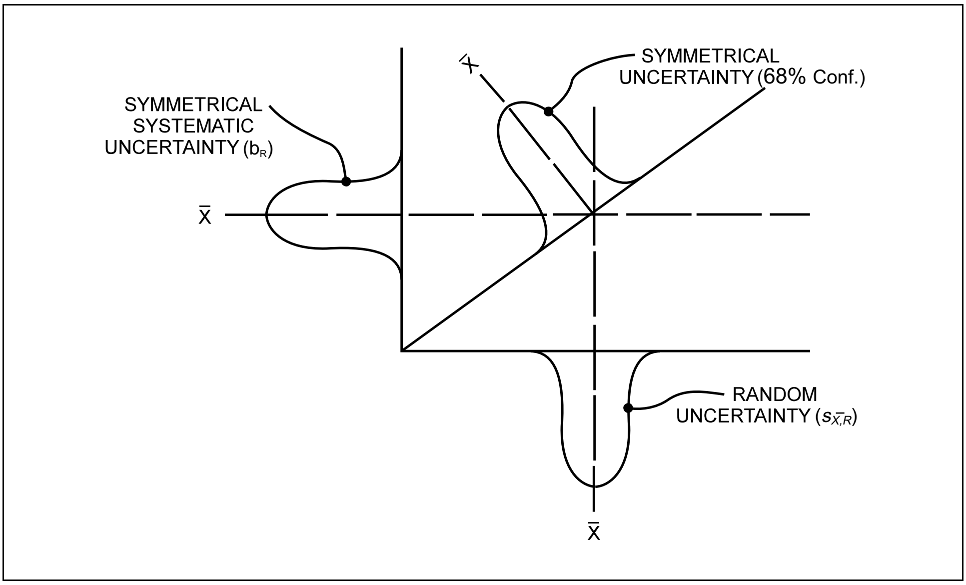

Figure 3-1. ~68% Confidence Symmetrical Uncertainty Interval

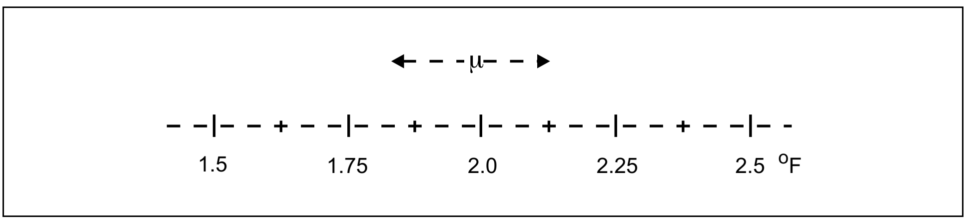

Figure 3-2. U_{95} Uncertainty Interval for μ = 2.00 ± 0.16

The U_{ASME} Uncertainty Interval (Symmetrical Systematic Uncertainty)

For the U_{ASME} uncertainty, the expression is:

$$U_{ASME} = \pm 2[(b_R)^2 + (s_{\bar{X},R})^2]^{1/2}$$
$$= \pm 2.00[(0.027)^2 + (0.073)^2]^{1/2}$$
$$= \pm 0.16\ °F$$

This is exactly the same as the U_{95} uncertainty. If there were fewer degrees of freedom, the resultant U_{95} uncertainty would be greater than the U_{ASME} because a larger Student's *t* would be needed.

The Choice of the Uncertainty Model

The uncertainty model should be chosen for the confidence desired. In some industries, such as aerospace, a great penalty is assessed if a wrong decision is made based on a test result. Therefore, in the past, the aerospace standard (Ref. 12) has utilized the U_{ADD} model, which provides 99% coverage of the true value. Also in the past, the steam turbine world preferred the 95% coverage of the U_{RSS} model.

Now and in the future, the choice of Student's t and the U_{95} model will afford the analyst the confidence of interest and still allow the use of the most robust uncertainty model, that is, U_{95}. This is true for the aerospace applications and the steam turbine applications just cited.

The choice is up to the user, but the model chosen must be reported.

3-6. The Calculation of Measurement Uncertainty (Nonsymmetrical Standard Systematic Uncertainties)

The first step here is the proper handling of the nonsymmetrical systematic standard uncertainties.

Combining Nonsymmetrical Systematic Standard Uncertainties

For most cases, the systematic standard uncertainties and the resulting uncertainty interval or measurement uncertainty will be symmetrical; that is, there is an equal risk of error in either the positive or the negative direction. However, there are times when the systematic standard uncertainty is not symmetrical. In these cases, something is known about the physics or engineering of the measurement that prompts a tendency for the errors to be either negative or positive.

An example of this kind of error might be a pressure reading from a transducer, which may tend to read low while trying to follow a ramp up in pressure. Another example might be a thermocouple in a process stream that, because of the laws of thermal conductivity, will always tend to read low when reading higher than ambient temperatures and high when reading lower than ambient temperatures.

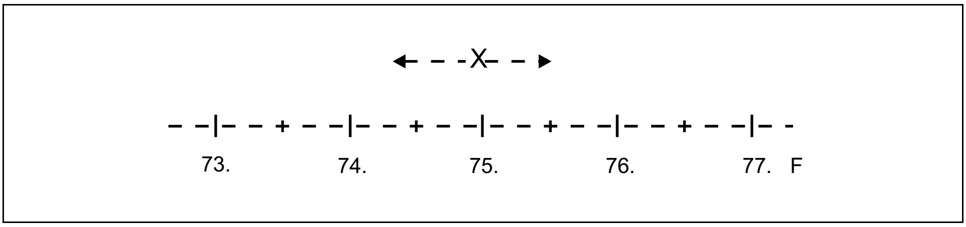

Figure 3-3. Nonsymmetrical Uncertainty Interval

Summarizing Uncertainty Categorization (Nonsymmetrical Systematic Standard Uncertainties)

Suppose that, of the uncertainties in Table 3-1, the calibration systematic standard uncertainty for the intermediate thermocouple reference was nonsymmetrical, in that the tendency was to read low for the higher than ambient calibration points. Given that the systematic standard uncertainty of ± 0.025 °F was nonsymmetrical as follows: –0.04 to +0.01 °F, it is noted that the min. to max. range of systematic standard uncertainty is still 0.05 °F. However, it is skewed negative. The uncertainty estimates then need to be reformatted from Table 3-1 into Table 3-2. This concept is shown schematically in Figure 3-4.

Table 3-2. Temperature Calibration Uncertainty Sources

Error Sources (i)	°F Systematic Standard Uncertainty (b_i^-)	(b_i^+)	°F Standard Deviation ($s_{X,i}$)	No. Data Points Avg'd. (N_i)*	°F Random Standard Uncertainty ($s_{\bar{X},i}$)	Degrees of Freedom (N – 1) (ν_i)
(1) Intermediate TC reference	-0.04	+0.01	0.056	1	0.056	29
(2) Ice reference junction	0.005	0.005	0.016	1	0.016	9
(3) Voltmeter readout	0.01	0.01	0.1	5	0.045	4

*See note on N_i following Table 3-1.

In Table 3-2 note the use of the notation b_i^- and b_i^+. The superscripts minus and plus indicate the negative and positive systematic uncertainties.

Since its systematic standard uncertainty is symmetrical, the uncertainty for the ice reference junction is reported as ±0.005 °F. When there are elemental systematic standard uncertainties that are nonsymmetrical, the proper handling of the summations requires that all systematic standard

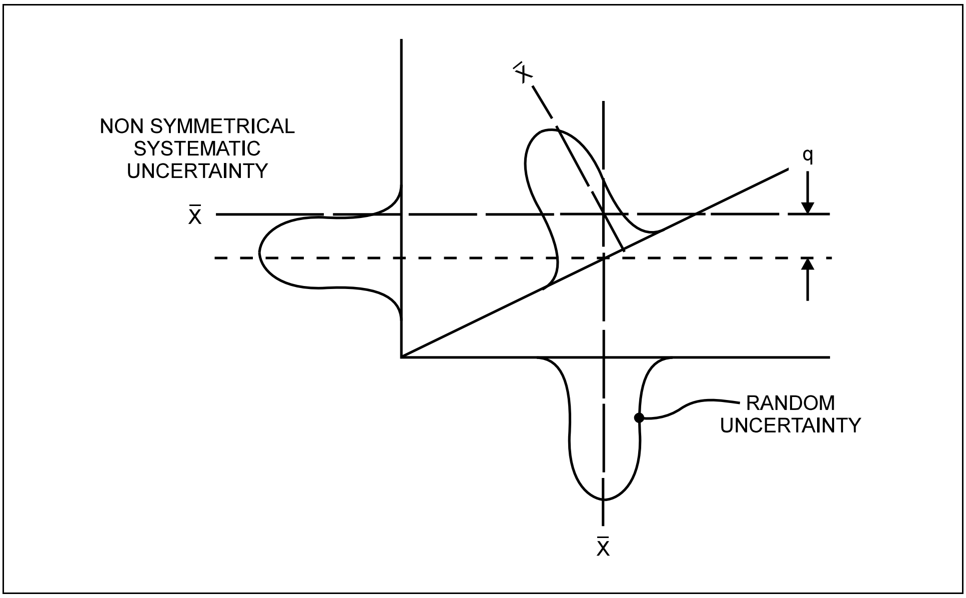

Figure 3-4. Uncertainty Interval for 74.24 °F ≤ $\overline{X}$ ≤ 75.36 °F

uncertainties be defined as minus and plus, or upper and lower limits. The systematic standard uncertainties for the ice reference junction therefore need to be noted as $b^{-} = 0.005$ °F and $b^{+} = 0.005$ °F, as shown in Table 3-2.

Systematic and random error sources and their uncertainties may be summarized as shown in Table 3-2.

Assuming the uncertainties noted are the uncertainties at the temperature calibration point of interest, the systematic standard uncertainties for the calibration are obtained as follows. Recalling Equation 3-7:

$$b_R = \left[\sum_{i=1}^{N}(b_i)^2\right]^{1/2} \tag{3-18}$$

The nonsymmetrical systematic standard uncertainty is computed using the methods in Ref. 13. This approach will be in the new U.S. national standard (Ref. 2). It proceeds as follows: Consider that for a normally distributed uncertainty source, its mean error may be represented as a displacement, q, from the central value such that a symmetrical systematic standard uncertainty interval may then be estimated about the displacement, $\overline{X} - q$. The symmetrical systematic standard uncertainty thus obtained (I love that expression) is then root-sum-squared with all the other symmetrical systematic standard uncertainties, a symmetrical total uncertainty interval is calculated, and then the displacement, q, is used to reinsert the nonsymmetry so the resulting final total uncertainty interval is nonsymmetrical. (Follow all that?)

If there is more than one nonsymmetrical elemental systematic standard uncertainty, several q values will be obtained and their use will create several symmetrical systematic standard uncertainties to root-sum-square for the final symmetrical systematic standard uncertainty. The several q values are then added algebraically and used to offset the resulting symmetrical total uncertainty to make it nonsymmetrical.

To see how this works, consider the following equations and the example of the uncertainties in Table 3-2. (Translation: These are the steps to follow for one nonsymmetrical systematic standard uncertainty source.)

First, the generalized equations are needed:

1. Given $s_{\overline{X}}$, b^+, and b^- (the nonsymmetrical systematic standard uncertainty) and the 68% confidence systematic uncertainty interval: $(\overline{X} - b^-)$ to $(\overline{X} + b^+)$

2. Let q equal the difference between the center of the nonsymmetrical systematic uncertainty interval and the average. That is:

$$q = [(\overline{X} + b^+) + (\overline{X} - b^-)]/2 - \overline{X} \tag{3-19}$$

3. Estimate a symmetrical b as one-half the nonsymmetrical systematic standard uncertainty interval:

$$b = [(\overline{X} + b^+) - (\overline{X} - b^-)]/2 \tag{3-20}$$

4. Complete the total uncertainty estimate for 30+ degrees of freedom:

$$U_{95} = \pm 2.00[(b_R)^2 + (s_{\overline{X},R})^2]^{1/2}$$

Recall that b_R and $s_{\overline{X},R}$ represent the root-sum-squares of the individual b's and $s_{\overline{X}}$'s. In this nonsymmetrical case, the b from Equation 3-19 is one of the b's root-sum-squared to obtain b_R.

5. Compute the 95% confidence uncertainty interval, I_{95}:

$$I_{95} = (\overline{X} + q) \pm U_{95} \tag{3-21}$$

6. Compute the nonsymmetrical, 95% confidence, total uncertainty interval, I, about the average:

$$I_{lower} = \overline{X} - (U_{95} - 2q) = \overline{X} - U^{-} \tag{3-22}$$

where I_{lower} is the lower limit of the nonsymmetrical 95% confidence total uncertainty interval.

$$I_{upper} = \overline{X} + (U_{95} + 2q) = \overline{X} + U^{+} \tag{3-23}$$

where I_{upper} is the upper limit of the nonsymmetrical 95% confidence total uncertainty interval.

Note that:

$$U^{-} = U_{95} - q \tag{3-24}$$

and that

$$U^{+} = U_{95} + q \tag{3-25}$$

Example 3-1

To see how this works, consider the data in Table 3-2. Going through the six steps shown above, we have the following:

1. For the first uncertainty source, the one with nonsymmetrical systematic uncertainty, $s_{\overline{X},1}$ = 0.056 °F, b_1^{+} = 0.01 °F, and b_1^{-} = 0.04 °F. Here assume the average for the test or experiment is 75 °F, $\overline{X}$ = 75 °F.

2. q_1 = [(75 + 0.01) + (75 – 0.04)]/2 – 75 = –0.015 (from Equation 3-18)

   Note the subscript 1 on q to denote the q from uncertainty source 1. Although there is only one nonsymmetrical systematic uncertainty in this example, were there more than one, the q would later need to be summed algebraically.

3. b_1 = [(75 + 0.01) – (75 – 0.04)]/2 = 0.025 (from Equation 3-19)

4. $U_{95} = \pm 2.00[(b_R)^2 + (s_{\overline{X},R})^2]^{1/2}$ (3-16)

$$b_R = [\Sigma(b_i)^2]^{1/2} = \pm[(0.025)^2 + (0.005)^2 + (0.001)^2]^{1/2} = 0.026\ °F$$

$$s_{\overline{X},R} = [\Sigma(s_{\overline{X},i})^2]^{1/2} = \pm[(0.056)^2 + (0.016)^2 + (0.045)^2]^{1/2} = 0.074\ °F$$

Therefore, assuming 30+ degrees of freedom:

U_{95} = ±2.00[(0.026)2 + (0.074)2]1/2 = ±0.16 °F

5. Compute the 95% confidence interval, I_{95} (from Equation 3-20):

I_{95} = [(75 + (–0.03)) ± 0.16] °F

6. The nonsymmetrical total uncertainty interval is then (from Equations 3-21 and 3-22):

I_{lower} = 75 – [0.16 – (–0.03)] = 74.87 °F
I_{upper} = 75 + [0.16 + (–0.03)] = 75.13 °F

Here, U^- = 0.16 – (–0.015) = 0.175 °F from Equation 3-23 and U^+ = 0.16 + (–0.015) = 0.145 °F (from Equation 3-24).

Therefore, the average and its uncertainty interval would be written as: (75 – 0.175) °F to (75 + 0.145) °F, or the 95% confidence uncertainty interval is 74.825 to 75.145 °F.

Figures 3-3 and 3-4 illustrate this nonsymmetrical uncertainty interval.

This is clearly a nonsymmetrical uncertainty interval about the average, 75 °F. It is a 95% confidence interval.

3-7. Common Uncertainty Model Summary

A common uncertainty model is summarized in Table 3-3 for the case of three categories and three uncertainty sources in each category. Note that here there are double subscripts under each uncertainty source, *i* and *j*, to designate an uncertainty source, *i*, in a category, *j*. No mention of Type A or B is shown here for clarity.

Table 3-3 shows clearly the summation and handling of the degrees of freedom. It can be used as a guide for as many categories as needed, with as many uncertainty sources as possible in each category, if grouping uncertainty sources in categories is desired. Remember, this is not necessary at all and some feel it is an unnecessary complication of the uncertainty estimation process. Table 3-3 is the uncertainty calculation for a single measurement or the measurement of a single parameter, such as temperature.

The b_R and $s_{\overline{X},R}$ are then used in the uncertainty calculation as appropriate.

If the result is not a simple parameter measurement, such as temperature or pressure, Table 3-3 is used for each parameter measurement used to calculate a result; for example, b_R, s_R, and v would be calculated for temperature, pressure, etc., and those values propagated into the final result uncertainty. This uncertainty propagation is covered in Unit 5.

If only the uncertainty of a single measurement result (e.g., temperature) is needed, the uncertainty is as shown in Table 3-3. To go further and determine the uncertainty for several results, the random uncertainty would again be divided by the square root of the number of results, M; that is, for multiple results:

$s_{\overline{X},R}/\sqrt{M}$ = the random error component of the uncertainty for several, M, results for one parameter such as temperature or pressure

M = the number of results to be averaged for a parameter

Table 3-3. Uncertainty Summary for Three Uncertainty Categories with Three Elemental Uncertainties Sources Each

Category	Systematic	Random	Degrees of Freedom
1	b_{11}, b_{21}, b_{31} } RSS = b_1	$s_{X,11}/\sqrt{N_{11}} = s_{\bar{X},11}$; $s_{X,21}/\sqrt{N_{21}} = s_{\bar{X},21}$; $s_{X,31}/\sqrt{N_{31}} = s_{\bar{X},31}$ } RSS = $s_{\bar{X},1}$	$\nu_{11}, \nu_{21}, \nu_{31}$ } W/S = ν_1
2	b_{12}, b_{22}, b_{32} } RSS = b_2	$s_{X,12}/\sqrt{N_{12}} = s_{\bar{X},12}$; $s_{X,22}/\sqrt{N_{22}} = s_{\bar{X},22}$; $s_{X,32}/\sqrt{N_{32}} = s_{\bar{X},32}$ } RSS = $s_{\bar{X},2}$	$\nu_{12}, \nu_{22}, \nu_{32}$ } W/S = ν_2
3	b_{13}, b_{23}, b_{33} } RSS = b_3	$s_{X,13}/\sqrt{N_{13}} = s_{\bar{X},13}$; $s_{X,23}/\sqrt{N_{23}} = s_{\bar{X},23}$; $s_{X,33}/\sqrt{N_{33}} = s_{\bar{X},33}$ } RSS = $s_{\bar{X},3}$	$\nu_{13}, \nu_{23}, \nu_{33}$ } W/S = ν_3
	↓ RSS = b_R, the systematic uncertainty	↓ RSS = $s_{\bar{X},R}$, the random uncertainty	↓ W/S = ν, the degrees of freedom of $s_{\bar{X},R}$

Note: RSS errors must have the same units. Propagate errors into result units before combining with RSS.
W/S = the Welch-Satterswaithe approximation for combining degrees of freedom. As with RSS, all the $s_{\bar{X}}$'s must have the same units.
RSS = the root-sum-square

Uncertainty Category	Systematic Uncertainty	Random Uncertainty	Degrees of Freedom	~Random Uncertainty of Source
1	b_1	$s_{\bar{X},1}$	ν_1	$t_{95}\ s_{\bar{X},1}$
2	b_2	$s_{\bar{X},2}$	ν_2	$t_{95}\ s_{\bar{X},2}$
3	b_3	$s_{\bar{X},3}$	ν_3	$t_{95}\ s_{\bar{X},3}$
	↓ RSS = b_R	↓ RSS = $s_{\bar{X},R}$	↓ W/S = ν_R	

Note: b_R = the systematic uncertainty of the result
$s_{\bar{X},R}$ = the random uncertainty of the result
ν_R = the degrees of freedom of the uncertainty, say, U_{95}
t_{95} = the Student's t for degrees of freedom for $s_{\bar{X},R}$

3-8. Detailed Experiment—Temperature—Without any Uncertainty Propagation

In this instructive example, we will first perform the uncertainty analysis emphasizing the random and systematic classifications of errors and uncertainties. The calculation will then be performed with the emphasis on the ISO Type A and Type B uncertainty classifications.

In this experiment, each member of a team will measure his/her body temperature using an ear thermometer by taking one measurement in their left ear and one measurement in their right ear. The team average will be compared to the 98.6 °F standard. In this experiment, we will explore the detailed calculations needed to estimate the uncertainty of the team average temperature and determine its use when comparing the team average body temperature to the usual standard of 98.6 °F.

Analysis Done with Random and Systematic Error and Uncertainty Classifications

1. **Test Data Acquisition**
   Each team member will measure their body temperature using an ear thermometer by taking one measurement in their left ear and one measurement in their right ear. The results will be recorded on a data sheet provided. The data obtained for the team members might look like Table 3-4.

Table 3-4. Acquired Data (Degrees F)

Team Member	Left	Right	Average
1	98.4	98.8	
2	97.8	97.6	
3	99.0	98.8	
4	98.7	98.9	
5	97.8	98.2	

The question now is: What is to be done with this data?

2. **Test Data and Random Standard Uncertainty Analysis**
The average of both measurements for each team member will then be calculated and the following results/data noted.

Table 3-5. Acquired Data (Degrees F) with Averages

Team Member	Left	Right	Average
1	98.4	98.8	98.6
2	97.8	97.6	97.7
3	99.0	98.8	98.9
4	98.7	98.9	98.8
5	97.8	98.2	98.0

A. Average (of the averages above) = 98.4 °F

B. s_X = 0.52 °F; ISO Type A. This is the standard deviation of the averages.

C. $s_{\overline{X}}$ = 0.52/(sqrt(5)) = 0.23 °F; ISO Type A. This is the random standard uncertainty.

D. *d.f.* = 4

B and C above are ISO Type A uncertainties because they result from statistical analysis. In this case, there is data to calculate a standard deviation.

3. **Determination of Systematic Standard Uncertainty**
Next, we need to consider the systematic error sources. Several are shown in Table 3-6. Once Table 3-6 is populated, it is necessary to calculate the systematic standard uncertainty, b_R, for the team's average, $b_R=[\Sigma(b_i)^2]^{1/2}$.

Table 3-6. Determination of Systematic Standard Uncertainty of the Result

System Uncertainty Sources	System Uncertainties (~95% Confidence)	ISO Type	System Standard Uncertainties (~68% Confidence)	Degrees of Freedom (*d.f.*)
Thermocouple Calibration	0.6	B	0.3	Infinity
Room Temp	0.1	B	0.05	Infinity
Battery Drain	0.2	B	0.1	Infinity
System Standard Uncertainty = $b_R = [(0.3)^2 + (0.05)^2 + (0.1)^2)^{1/2} = 0.32$ °F				

Here the degrees of freedom would normally be determined using the Welch-Satterthwaite approximation. For this case, it may be done by inspection; it is infinity.

4. **Determination of Combined and Expanded Uncertainties**
Now the combined uncertainty is calculated as the root-sum-square of the systematic standard uncertainty (0.32 °F) and the random standard uncertainty (0.23 °F).

The combined uncertainty is then:

$$U_c = [(0.32\ °F)^2 + (0.23\ °F)^2]^{1/2} = 0.39\ °F \qquad (3\text{-}26)$$

This combined uncertainty is then multiplied by Student's t once the combined degrees of freedom are calculated by the Welch-Satterthwaite approximation.

The degrees of freedom computed with the Welch-Satterthwaite approximation are as follows:

$$d.f. = \{[(0.32)^2 + (0.23)^2]^2\} \ / \ \{(0.32)^4/(\infty) + (0.23)^4/4\} = 34 \qquad (3\text{-}27)$$

Note that there are 4 degrees of freedom for the random standard uncertainty in analysis step 2, result D.

With 147 degrees of freedom, Student's t is equal to 2.00.

Therefore, we have for the expanded 95% confidence uncertainty:

$$U_{95} = (+/- \ 2.00 * 0.39\ °F) = +/- \ (0.78\ °F) \qquad (3\text{-}28)$$

This means that our average of 98.0 °F, plus or minus 0.78 °F, should include the true value of 98.6 °F and it does.

Analysis Done with ISO Type A and Type B Classification of Errors and Uncertainties

We will not repeat the analysis but will apply the ISO Type A and Type B classifications of standard uncertainties. Recall that Type A standard uncertainties arise from statistical analysis, which usually means there is data available to calculate a standard deviation to describe that standard uncertainty. Type B standard uncertainties are obtained in a nonstatistical way, such as using a manufacturer's specifications for instruments, uncertainties provided by a calibration laboratory, or possibly engineering estimates.

For this experiment, we have already identified the Type A and Type B standard uncertainties. We now need to group them and perform the calculations.

The combined, root-sum-square value for the Type A standard uncertainties is determined with the following equation.

$$u_A = \left[\sum_{i=1}^{N_A} (u_{A,i})^2 \right]^{1/2} = \_\_\_\_\_ \qquad d.f. = \_\_\_\_\_ \tag{3-29}$$

In this example analysis, there is only one Type A uncertainty and it is in analysis step 2, result C: 0.23 °F with 4 degrees of freedom. Hence:

$$u_A = \left[\sum_{i=1}^{N_A} (u_{A,i})^2 \right]^{1/2} = 0.23 \qquad d.f. = 4 \tag{3-30}$$

In this example, this is a trivial result.

To determine the Type B standard uncertainty, we use the following equation:

$$u_B = \left[\sum_{i=1}^{N_B} (u_{B,i})^2 \right]^{1/2} = \_\_\_\_\_ \qquad d.f. = \_\_\_\_\_ \tag{3-31}$$

Here we actually have three sources of Type B standard uncertainties. They are shown as combined in section 3 above and equal 0.32 °F. Hence:

$$u_B = [(0.3)^2 + (0.05)^2 + (0.1)^2)^{1/2} = 0.32\ °\text{F} \qquad d.f. = \infty \tag{3-32}$$

The combined standard uncertainty is then:

$$U_c = [(0.32\ °\text{F})^2 + (0.23\ °\text{F})^2]^{1/2} = 0.39\ °\text{F} \tag{3-33}$$

which is identical to the combined uncertainty obtained in analysis step 4 above.

To compute the expanded uncertainty, we first need to compute the combined degrees of freedom. This is identical to that computed in analysis step 4 above, or 34.

The expanded uncertainty is also identical to that in analysis step 4 above, or +/- 0.78 °F.

It is clear, even from this simple example, that the ASME approach using random and systematic classifications of standard uncertainties yields the identical expanded uncertainty as the ISO approach using Type A and Type B classifications of standard uncertainties. For a more detailed example of this equivalency, see Section 4-9, "Interplay of Random and Systematic Effects on an Uncertainty Analysis," of this text.

3-9. Summary

This unit has presented the basics of the error and elemental uncertainty categorization process and uncertainty statement. It is now possible for the student to calculate the measurement uncertainty for simple measurement systems.

3-10. References

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3-11. Exercises

3-1. Systematic Uncertainty Problems

Consider the following list of systematic standard uncertainties: 1, 2, 3, 10.

a. What would you get after combining them to obtain the systematic standard uncertainty of the test result?

1) 10.68 2) 106.00 3) 13.74 4) ________

b. Same as (a), but with systematic standard uncertainties of: 1, 2, 3, 10, 11, 12.

1) 19.47 2) 39.00 3) 36.74 4) ________

c. What is learned about the impact of large uncertainties?

3-2. Random Uncertainty Problems

Consider the following random standard uncertainties for two uncertainty sources:

$s_{\overline{X},1} = 1.0\%$ $\quad v_1 = 10$ degrees of freedom

$s_{\overline{X},2} = 2.0\%$ $\quad v_2 = 20$ degrees of freedom

a. What is the combined random standard uncertainty, $s_{\overline{X},R} = (s^2_{\overline{X},1} + s^2_{\overline{X},2})^{1/2}$?

b. What are the equivalent degrees of freedom? (Note: Use the Welch-Satterthwaite approximation.)

c. What is $t_{95} * s_{\overline{X},R}$? (This is the random standard uncertainty component of the result.)

3-3. Combining Degrees of Freedom Problem

Write the formula for the standard deviation of the result, $s_{\overline{X},R}$, that combines the individual random uncertainties, $s_{\overline{X},i}$.

3-4. Truncating Degrees of Freedom Problem

Why is truncating the fractional degrees of freedom obtained with the Welch-Satterthwaite approximation considered to be conservative?

3-5. Nonsymmetrical Uncertainty Interval Problem

Given $s_{\overline{X}} = 4$, $b^+ = 1.5$, $\text{b}^- = 3.5$ and $\overline{X} = 26$, calculate the U_{95} (95% confidence) nonsymmetrical uncertainty interval (the upper and lower nonsymmetrical uncertainties).

3-6. Scale and Truth Problem

Consider every possible uncertainty source for a measurement process that consists of daily weighings on your bathroom scale in preparation for your annual physical examination when you will be weighed on your doctor's scale. Categorize the uncertainty sources into systematic and random.

a. What is truth in this process?

b. How would you calibrate your scale?

c. How would you estimate the random uncertainty?

d. Make your best estimate of the systematic and random uncertainties of a typical bathroom scale.